Universal algebra + Transformations

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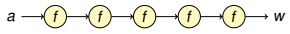
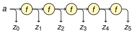
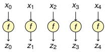
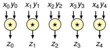
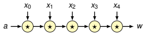
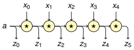
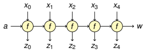
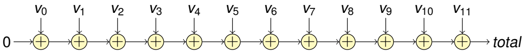
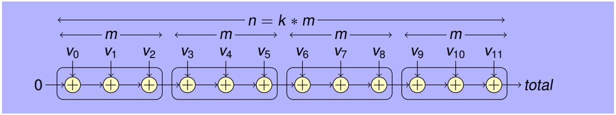
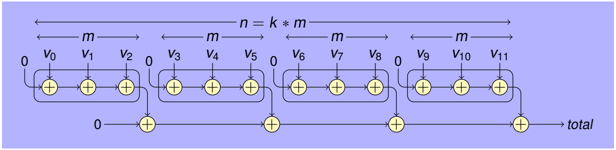
name	*structure*	*Haskell*	*Math*
itn		$w = itn\ f\ a\ n$	$w = f^n\ a$
itnscanl		$zs = itnscanl\ f\ a\ n$	
map		$zs = map\ f\ xs$	$zs = \widehat{f}\ xs$
zipWith		$zs = zipWith\ (\star)\ xs\ ys$	$zs = xs \mathbin{\widehat{\star}} ys$
foldl		$w = foldl\ (\star)\ a\ xs$	$w = a \circledast xs$
scanl		$zs = scanl\ (\star)\ a\ xs$	
mapAccumL		$(w, zs) = mapAccumL\ f\ a\ xs$	

Transformations (D2.1)

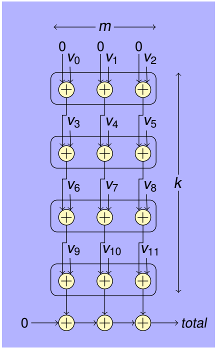




No properties needed



Associative, Zero



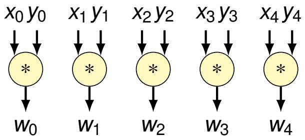
Associative, Commutative, Zero

Dot product

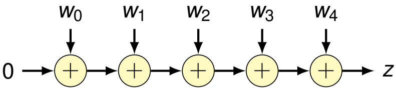
$$\vec{x} \bullet \vec{y} \;=\; \sum_{i=0}^{n-1} x_i y_i \;=\; x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1}$$

$$\vec{x} \bullet \vec{y} \;=\; \sum_{i=0}^{n-1} x_i y_i \;=\; x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1}$$

$$\vec{w} \;=\; \vec{x} \;\widehat{*}\; \vec{y}$$



$$z \;=\; 0 \;\bigoplus\; \vec{w}$$



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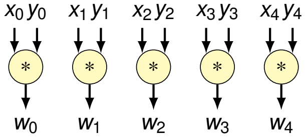
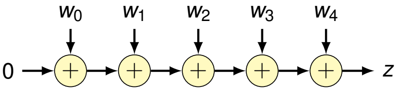
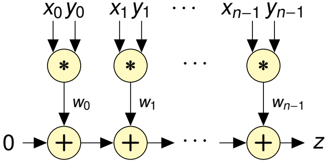


$$z \;=\; 0 \;\bigoplus\; \vec{w}$$



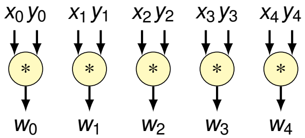
$$\vec{x} \bullet \vec{y} \;=\; 0 \;\bigoplus\; (\vec{x} \;\widehat{*}\; \vec{y})$$



$$\vec{x} \bullet \vec{y} \;=\; \sum_{i=0}^{n-1} x_i y_i \;=\; x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1}$$

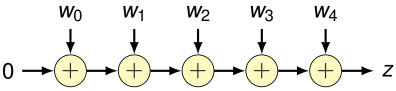
$$\vec{w} \;=\; \vec{x} \;\widehat{*}\; \vec{y}$$

$$\mathit{ws} \;=\; \mathit{zipWith}\ (*)\ \mathit{xs}\ \mathit{ys}$$



$$z \;=\; 0 \;\oplus\; \vec{w}$$

$$z \;=\; \mathit{foldl}\ (+)\ 0\ \mathit{ws}$$

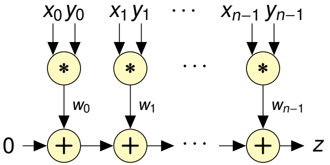


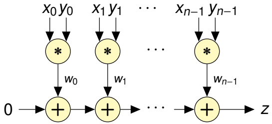
$$\vec{x} \bullet \vec{y} \;=\; 0 \;\oplus\; (\vec{x} \;\widehat{*}\; \vec{y})$$

$$\mathit{xs} \mathbin{\langle\cdot\rangle} \mathit{ys} \;=\; \mathit{foldl}\ (+)\ 0\ \mathit{ws}$$

where

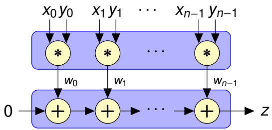
$$\mathit{ws} = \mathit{zipWith}\ (*)\ \mathit{xs}\ \mathit{ys}$$



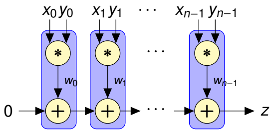


$$xs \bullet ys \;=\; foldl\ (+)\ 0\ \$\ \ zipWith\ (*)\ xs\ ys$$

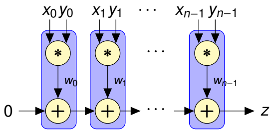
Transformation on dot product



$$xs \bullet ys \;=\; \mathit{foldl}\ (+)\ 0\ \$\ \mathit{zipWith}\ (*)\ xs\ ys$$



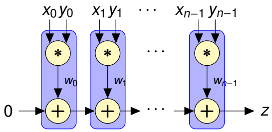
$$xs \bullet ys \ = \ foldl\ (+)\ 0\ \$\ zipWith\ (*)\ xs\ ys$$



$$xs \bullet ys \;=\; foldl\ (+)\ 0\ \$\ zipWith\ (*)\ xs\ ys$$

$$(\star) \triangleleft f = \lambda x\, y.\ x \star f(y)$$

$$f \triangleright (\star) = \lambda x\, y.\ f(x) \star y$$



$$xs \bullet ys \;=\; foldl\ (+)\ 0\ \$\ zipWith\ (*)\ xs\ ys$$

$$(\star) \triangleleft f = \lambda x\, y.\ x \star f(y)$$

$$f \triangleright (\star) = \lambda x\, y.\ f(x) \star y$$

$$
\begin{aligned}
xs \bullet ys \;&= foldl\ op\ 0\ \$\ zip\ xs\ ys \\
&\textbf{where} \\
&\quad f\,(x, y) \quad = x * y \\
&\quad op \qquad\;\; = (+) \triangleleft f
\end{aligned}
$$

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \;=\; x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\vec{x} \bullet \vec{y} \;=\; 0 \;\bigoplus\; \left(\vec{x} \,\widehat{*}\, \vec{y}\right)$$

$$xs \diamond ys \;=\; foldl\;(+)\;0\;\$\;zipWith\;(*)\;xs\;ys$$

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \\ 41 & 42 & 43 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 74 \\ 134 \\ 194 \\ 254 \end{pmatrix}$$

$$\begin{aligned} \textbf{type}\ V\ a &= [a] \\ \textbf{type}\ M\ a &= [V\ a] \end{aligned}$$

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

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type *V a* = [*a*]
type *M a* = [*V a*]

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

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Matrix-vector multiplication

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

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$$\begin{aligned} \textbf{type } V\ a &= [a] \\ \textbf{type } M\ a &= [V\ a] \end{aligned}$$

$$A \times \vec{y} = \widehat{\bullet\vec{y}}\ A$$

Matrix-vector multiplication

$$(x_1, x_2, x_3) \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

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$$\begin{aligned} \textbf{type}\ V\ a &= [a] \\ \textbf{type}\ M\ a &= [V\ a] \end{aligned}$$

$$A \times \vec{y} = \widehat{\bullet\vec{y}}\ A$$

$$xss \lessdot ys = map\ (\diamond\, ys)\ xss$$

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \\ 41 & 42 & 43 \end{pmatrix} \star \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} = \begin{pmatrix} 74 & 182 & 290 & 398 \\ 134 & 332 & 530 & 728 \\ 194 & 482 & 770 & 1058 \\ 254 & 632 & 1010 & 1388 \end{pmatrix}$$

Matrix-matrix multiplication

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \\ 41 & 42 & 43 \end{pmatrix} \star \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} = \begin{pmatrix} 74 & 182 & 290 & 398 \\ 134 & 332 & 530 & 728 \\ 194 & 482 & 770 & 1058 \\ 254 & 632 & 1010 & 1388 \end{pmatrix}$$

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$$A \star B = (\widehat{A\times}\ B^T)^T$$

Matrix-matrix multiplication

$$\begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \\ 41 & 42 & 43 \end{pmatrix} \star \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix} = \begin{pmatrix} 74 & 182 & 290 & 398 \\ 134 & 332 & 530 & 728 \\ 194 & 482 & 770 & 1058 \\ 254 & 632 & 1010 & 1388 \end{pmatrix}$$

$$A \star B = (\widehat{A \times}\ B^T)^T$$

$$xss \ \lessdot\!\gtrdot\ yss = transpose\ \$\ map\ (xss \lessdot\cdot)\ \$\ transpose\ yss$$

Matrix-matrix multiplication

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 \\ 5 & 6 \\ 6 & 7 \end{pmatrix} & \begin{pmatrix} 6 & 7 \\ 7 & 8 \\ 8 & 9 \end{pmatrix} \end{pmatrix} \star \begin{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} & \begin{pmatrix} 6 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{pmatrix} = \quad \cdots$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 \\ 5 & 6 \\ 6 & 7 \end{pmatrix} & \begin{pmatrix} 6 & 7 \\ 7 & 8 \\ 8 & 9 \end{pmatrix} \end{pmatrix} \star \begin{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} & \begin{pmatrix} 6 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 70 & 60 & 50 \\ 100 & 86 & 72 \\ 130 & 112 & 94 \end{pmatrix} & \begin{pmatrix} 40 & 30 & 20 \\ 58 & 44 & 30 \\ 76 & 58 & 40 \end{pmatrix} \\ \begin{pmatrix} 160 & 138 & 116 \\ 190 & 164 & 138 \\ 220 & 190 & 160 \end{pmatrix} & \begin{pmatrix} 94 & 72 & 50 \\ 112 & 86 & 60 \\ 130 & 100 & 70 \end{pmatrix} \end{pmatrix}$$

Matrix-matrix multiplication

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 \\ 5 & 6 \\ 6 & 7 \end{pmatrix} & \begin{pmatrix} 6 & 7 \\ 7 & 8 \\ 8 & 9 \end{pmatrix} \end{pmatrix} \star \begin{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} & \begin{pmatrix} 6 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 70 & 60 & 50 \\ 100 & 86 & 72 \\ 130 & 112 & 94 \end{pmatrix} & \begin{pmatrix} 40 & 30 & 20 \\ 58 & 44 & 30 \\ 76 & 58 & 40 \end{pmatrix} \\ \begin{pmatrix} 160 & 138 & 116 \\ 190 & 164 & 138 \\ 220 & 190 & 160 \end{pmatrix} & \begin{pmatrix} 94 & 72 & 50 \\ 112 & 86 & 60 \\ 130 & 100 & 70 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \star \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} = \quad \cdots$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 \\ 8 & 7 & 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 70 & 60 & 50 & 40 & 30 & 20 \\ 100 & 86 & 72 & 58 & 44 & 30 \\ 130 & 112 & 94 & 76 & 58 & 40 \\ 160 & 138 & 116 & 94 & 72 & 50 \\ 190 & 164 & 138 & 112 & 86 & 60 \\ 220 & 190 & 160 & 130 & 100 & 70 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \\ \begin{pmatrix} 4 & 5 \\ 5 & 6 \\ 6 & 7 \end{pmatrix} & \begin{pmatrix} 6 & 7 \\ 7 & 8 \\ 8 & 9 \end{pmatrix} \end{pmatrix} \star \begin{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} & \begin{pmatrix} 6 & 5 & 4 \\ 5 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} & \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 70 & 60 & 50 \\ 100 & 86 & 72 \\ 130 & 112 & 94 \end{pmatrix} & \begin{pmatrix} 40 & 30 & 20 \\ 58 & 44 & 30 \\ 76 & 58 & 40 \end{pmatrix} \\ \begin{pmatrix} 160 & 138 & 116 \\ 190 & 164 & 138 \\ 220 & 190 & 160 \end{pmatrix} & \begin{pmatrix} 94 & 72 & 50 \\ 112 & 86 & 60 \\ 130 & 100 & 70 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix} \star \begin{pmatrix} 9 & 8 & 7 \\ 8 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} \star \begin{pmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 25 & 22 & 19 \\ 42 & 37 & 32 \\ 59 & 52 & 45 \end{pmatrix} + \begin{pmatrix} 45 & 38 & 31 \\ 58 & 49 & 40 \\ 71 & 60 & 49 \end{pmatrix}$$

Components:

- Carrier set(s) $a, b, \ldots$
- n-ary operations
- neutral elements
- properties (associativity, commutativity, ...)

Variants:

- Set
- *One* binary operation: Magma, Semigroup, Monoid, ...
- *Two* binary oprations: Semi-ring, Ring, Field, ...
- *Two or more* carrier sets: Modules, Vector space, ...

Nice overview: `http://en.wikipedia.org/wiki/Algebraic_structure`

Basics:

- *Carrier set*: a
- *Binary operations*: $+$, $*$
- *Neutral elements*: 0, 1
- *Properties*: associative, $*$ distributive over $+$, $0 * x = 0$, ...

Semi-Ring

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Definitions for vectors/matrices over a:

- *Vector addition*
- *Scalar multiplication*
- *Dot product*
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Semi-Ring

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Examples:

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- $\mathbb{B}$ with $\vee$, $\wedge$, *False*, *True*
- Max-plus algebra ($\mathbb{R} \cup \{-\infty\}$, $\otimes$, $\oplus$, 0, $-\infty$)
- **Matrices** over semiring *a*, with element-wise addition, matrix product, 0-matrix, identity matrix

Annotations:

- assume a specific algebraic structure (in this case semi-ring)
- exploit algebraic laws for partitiong/parallelization of code

Question: how can we be sure that an annotation really expresses the computational structure in the code? (HiPEAC workshop)

$\Longrightarrow$ Executable formalism close enough to annotation formalism for testing: Haskell

Higher order *stencil* functions:

- $stencil1D\ f\ w\ xs$
- $stencil2D\ f\ (v, w)\ xss$

One-dimensional heatflow:

$$stencil1D\ \ f\ \ 3\ \ ts$$

where

$$f\ [t_0, t_1, t_2]\ =\ t_1 + k * (t_0 - 2 * t_1 + t_2)$$

?